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Prequantization and KMS Structures

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The properties of the representations of the canonical commutation relations, obtained in the prequantization program, are investigated with special attention to the relevance of the KMS structures in this context. In particular, we show how these structures provide a natural way to pass from the prequantization representation of the CCR to the Schrödinger representation.

1. INTRODUCTION AND RESULTS

The prequantization program was stated in van Hove (1951) as an exploration of the possible links between the invariance groups of classical and quantum mechanics, respectively; it was later expanded by Souriau (1966) and Kostant (1970) to a systematic attempt at the formulation of general quantization rules allowing one to pass from classical to quantum mechanics; for a comprehensive presentation, see Simms and Woodhouse (1976), Guillemin and Sternberg (1977), Sniatycki (1980).

The first step in this program is to find a canonical, injective, linear representation of the Lie algebra $\mathcal{C}$ of suitably smooth functions on a symplectic manifold $\mathfrak{M}$, by self-adjoint operators acting on a Hilbert space $\mathcal{H}$, in such a manner that the classical Poisson bracket is transformed into the commutator, familiar in quantum theory, between the corresponding operators.

In particular when the configuration space of the classical system is R^n, $\mathfrak{M} \simeq R^{2n}$ is its phase space, namely, the cotangent bundle of R^n; for typographical simplicity, we write explicit expressions for $n=1$, since the generalization of the forthcoming remarks to any positive, finite integer n is straightforward. $\mathfrak{M}$ comes then naturally equipped with a symplectic form, namely, $\omega = dp \wedge dq$. The Hamiltonian vector field X_f is then associated to

$f \in \mathcal{C}$ by $X_f \lrcorner \omega = -df$; and the Poisson bracket between f and g in $\mathcal{C}$ is

$$\{f, g\} = -\omega(X_f, X_g) = \partial_q f \partial_p g - \partial_p f \partial_q g$$

To the one-form $\theta = p\, dq$, with $d\theta = \omega$ and thus $d\omega = 0$, one associates the covariant derivative ∇_X and one verifies that the map $\mathcal{P}: f \mapsto \hat{f}$, defined by:

$$\begin{aligned} \hat{f} &= -i\kappa \nabla_{X_f} + f \\ &= i\kappa(\partial_q f)\partial_p - i\kappa(\partial_p f)\partial_q - p(\partial_p f) + f \end{aligned}$$

induces in the space $\mathcal{H}$ of square integrable functions on $\mathfrak{M}$, namely, $\mathcal{L}^2(R^{2n}, dp\, dq)$, a representation of $\mathcal{C}$ satisfying

$$\mathcal{P}(\alpha f + \beta g) = \alpha \hat{f} + \beta \hat{g}$$

$$\mathcal{P}(\{f, g\}) = -i[\hat{f}, \hat{g}]/\kappa$$

$$\mathcal{P}(1) = I$$

where κ is a positive constant, adjustable to be equal to Planck's $h/2\pi$. There is then a linear subspace $\mathcal{D}$, dense in $\mathcal{H}$, on which

$$\hat{p} = -i\kappa\partial_q \quad \text{and} \quad \hat{q} = q + i\kappa\partial_p$$

are essentially self-adjoint and satisfy the familiar commutation relation $[\hat{p}, \hat{q}] = -i\kappa I$. This representation of the canonical commutation relations is not irreducible and thus does not quite realize the so-called Dirac program; hence the name *prequantization* for the map $\mathcal{P}$ just described. It is interesting to note with Streater (1966) that $\mathcal{H}$ harbors another representation of the CCR, namely,

$$\tilde{p} = -i\kappa\partial_p \quad \text{and} \quad \tilde{q} = p + i\kappa\partial_q$$

which commutes with both $\hat{p}$ and $\hat{q}$. The existence of this companion representation will appear as a natural consequence of the KMS structures to be delineated in the present paper. For the time being we only recall that the failure to obtain an irreducible representation is a manifestation of a general no-go theorem of which Chernoff gave an enlightening proof, the gist of which can be found in Abraham and Marsden (1978).

The purpose of the present paper is to study the von Neumann algebra $\mathcal{W}$ generated by $\hat{p}$ and $\hat{q}$. To avoid dealing with unbounded operators, and the attendant domain questions, we note that this algebra is obtained as the closure in the ultraweak topology of the linear span in $\mathfrak{B}(\mathfrak{H})$ of the unitary operators

$$\{W(z)|z=a+ib\in C\}$$

defined by

$$W(z)=U(a)V(b)\exp(-iab/2\kappa)$$

where

$$U(a)=\exp(-i\hat{p}a/\kappa)$$

$$V(b)=\exp(-i\hat{q}b/\kappa)$$

so that

$$W(z_1)W(z_2)=W(z_1+z_2)\exp\left[i\operatorname{Im}(z_1^*z_2)/2\kappa\right]$$

Our first result is the following:

Proposition 1. (i) $\mathfrak{W}$ is a factor; (ii) for every faithful normal state ϕ on $\mathfrak{W}$, there exists $\Phi\in\mathfrak{H}$ such that: (a) $\langle\phi; W\rangle=(W\Phi,\Phi)$ for every $W\in\mathfrak{W}$, and (b) $[\mathfrak{W}\Phi]=\mathfrak{H}=[\mathfrak{W}'\Phi]$.

The first part of this proposition reads $\mathfrak{W}\cap\mathfrak{W}'=C\cdot I$, i.e., the only observables in $\mathfrak{W}$ which commute with $\hat{p}$ and $\hat{q}$ are the scalar multiples of the trivial observable I. This is reminiscent of the situation encountered in the usual (irreducible) Schrödinger representation. This should nevertheless be constrasted with the second part of the proposition, which states that the commutant $\mathfrak{W}'$ of $\mathfrak{W}$ is rather large since it admits a cyclic vector. For completeness we recall (see, e.g., Dixmier, 1957) that a state ϕ on a von Neumann algebra $\mathfrak{W}$ is a linear, positive functional of $\mathfrak{W}$, normalized by $\langle\phi; I\rangle=1$; ϕ is said to be normal if it is continuous in the ultraweak topology; this is equivalent to the requirement that for every family $\{F_\iota|\iota\in I\}$ of mutually orthogonal projectors $F_\iota\in\mathfrak{W}:\Sigma_\iota\langle\phi; F_\iota\rangle=\langle\phi;\Sigma_\iota F_\iota\rangle$. Hence the requirement that a state be normal is the straightforward generalization, to the noncommutative context, of the requirement that a measure be completely additive. Finally a state ϕ on $\mathfrak{W}$ is said to be faithful if $\langle\phi; A^*A\rangle=0$ and $A\in\mathfrak{W}$ imply $A=0$.

As a consequence of the second part of Proposition 1, we obtain that every normal state ψ on $\mathfrak{W}$ can be described by a vector $\Psi\in\mathfrak{H}$ through the formula $\langle\psi; W\rangle=(W\Psi,\Psi)$ for all $W\in\mathfrak{W}$. This again is in sharp contradistinction with the Schrödinger representation where a general density matrix induces a state which cannot be so expressed. Consequently we should guard against interpreting vectors in $\mathfrak{H}$ as pure states on $\mathfrak{W}$.

The fact that the vector Φ of Proposition 1 is both cyclic (i.e., $[\mathcal{W}\Phi]=\mathcal{H}$) and separating (i.e., $A\Phi=0$ and $A\in\mathcal{W}$ imply $A=0$) for $\mathcal{W}$ (the latter being a consequence of $[\mathcal{W}'\Phi]=\mathcal{H}$) is central to the purpose of this paper. The Tomita–Takesaki theory (see Takesaki, 1970) indeed then asserts that the map

$$S_\Phi: W\Phi\in\mathcal{W}\Phi\subseteq\mathcal{H}\mapsto W^*\Phi\in\mathcal{W}\Phi\subseteq\mathcal{H}$$

is closable; its closure is a densely defined, closed antilinear map (denoted again by S_Φ) from which one can define two operators $\Delta_\Phi=S_\Phi^*S_\Phi$ and $J_\Phi=S_\Phi\Delta_\Phi^{-1/2}$ with Δ_Φ self-adjoint and J_Φ antiunitary such that $J_\Phi^2=I$; $J_\Phi\Phi=\Phi$; $W\mapsto J_\Phi W J_\Phi$ induces an antilinear isomorphism from $\mathcal{W}$ onto $\mathcal{W}'$; and

$$W\mapsto\alpha_\phi(t)[W]=\Delta_\Phi^{-it/\beta}W\Delta_\Phi^{it/\beta}$$

induces a continuous one-parameter group $\alpha_\phi(R)$ of automorphisms of $\mathcal{W}$. Moreover $\alpha_\phi(R)$ is the only one-parameter group of automorphisms of $\mathcal{W}$ such that for any pair (A, B) of elements of $\mathcal{W}$ there exists a function $F_{A,B}$ analytic in the strip $\{z\,|\operatorname{Im} z\in(0,\beta)\}$ and continuous on its boundaries, such that for all $t\in R$

$$F_{A,B}(t)=\langle\phi; A\alpha_\phi(t)[B]\rangle$$

and

$$F_{A,B}(t+i\beta)=\langle\phi; \alpha_\phi(t)[B]A\rangle$$

This is the KMS condition, named after Kubo (1957) and Martin and Schwinger (1959). Its relevance in the algebraic formulation of quantum statistical mechanics was established by Haag, Hugenholtz, and Winnink (1967), while some of the attendant structure had already been isolated by Araki and Woods (1963); the general mathematical theory was made available by Takesaki (1970).

For any faithful normal state ϕ on $\mathcal{W}$ (as in Proposition 1), the centralizer $\mathfrak{M}_\phi$ of $\mathcal{W}$ with respect to ϕ is the von Neumann algebra

$$\mathfrak{M}_\phi=\{A\in\mathcal{W}\,|\langle\phi;[A,W]\rangle=0\ \forall W\in\mathcal{W}\}$$

$$=\{A\in\mathcal{W}\,|\,\alpha_\phi(t)[A]=A\ \forall t\in R\}$$

It is then known (see Takesaki, 1972) that there exists a normal conditional expectation $\mathcal{E}$ from $\mathcal{W}$ onto $\mathfrak{M}_\phi$ such that $\phi\circ\mathcal{E}=\phi$. For the purpose of interpreting the physical meaning of $\mathfrak{M}_\phi$, we consider an arbitrary partition $\mathcal{F}=\{F_k\}$ of the identity I in mutually orthogonal projectors F_k in $\mathcal{W}$, and define for every k the state

$$\phi_k: W\in\mathcal{W}\mapsto\langle\phi;\, F_kWF_k\rangle/\langle\phi;\, F_k\rangle\in C$$

Following von Neumann (1932), the effect of the measurement of $\mathcal{F}$ is to change the state ϕ into the state

$$\mathcal{F}[\phi]=\sum_k\lambda_k\phi_k \qquad \text{with} \qquad \lambda_k=\langle\phi;\, F_k\rangle$$

$\mathfrak{M}_\phi$ does then coincide (Emch, 1976) with the von Neumann algebra generated by all partitions $\mathcal{F}$ that satisfy $\mathcal{F}[\phi]=\phi$.

Our next result delineates the structure of the centralizer $\mathfrak{M}_\phi$ of the von Neumann algebra $\mathcal{W}$ associated to the prequantization map $\mathcal{P}$, when ϕ is a faithful, nondegenerate, normal state on $\mathcal{W}$. We first recall that a von Neumann subalgebra $\mathfrak{N}$ of $\mathcal{W}$ is said to be: Abelian if $\mathfrak{N}\subseteq\mathfrak{N}'$; maximal Abelian in $\mathcal{W}$ if $\mathfrak{N}=\mathcal{W}\cap\mathfrak{N}'$; and atomic (or discrete) if the collection $\mathcal{F}$ of all projectors in $\mathfrak{N}$ is an atomic lattice under the natural ordering $E\preccurlyeq F$ if $E\mathcal{H}\subseteq F\mathcal{H}$.

Proposition 2. (i) $\mathfrak{M}_\phi$ is an atomic, maximal Abelian von Neumann subalgebra of $\mathcal{W}$; (ii) up to an additive constant αI, there exists a unique self-adjoint element $H_\phi\in\mathfrak{M}_\phi$ such that

$$\alpha_\phi(t)[W]=\exp(iH_\phi t)W\exp(-iH_\phi t)$$

for all $t\in R$ and all $W\in\mathcal{W}$; (iii) with H_ϕ as in (ii) we have

(a) $-(1/\beta)\ln\Delta_\Phi=H_\Phi-J_\Phi H_\phi J_\Phi$
(b) $-(1/\beta)\ln\Delta_\Phi\cdot W\Phi=[H_\phi,W]\Phi\ \ \forall W\in\mathcal{W}$
(c) $\mathfrak{M}_\phi=\{H_\phi\}''$;
(d) the spectrum $\mathrm{Sp}(H_\phi)=\{\varepsilon_i|i\in Z^+\}$ of H_ϕ is discrete, simple in $\mathcal{W}$, and satisfies the condition $\Sigma_i\exp(-\beta\varepsilon_i)$ finite.

We next indicate how the structures described in these two propositions can be exploited to recover the irreducible Schrödinger realization in which one usually describes the quantum mechanics of a system with finitely many degrees of freedom. As in Souriau (1966) and Kostant (1970) our procedure recognizes the role played by some particular choice of a complete set of mutually compatible observables.

Proposition 3. Let $\mathcal{Q}$ be any atomic, maximal Abelian von Neumann subalgebra of $\mathcal{W}$ and f be any normal pure state on $\mathcal{Q}$. Then: (i) f admits a unique extension to a state ψ on $\mathcal{W}$, and this state is normal and pure on $\mathcal{W}$;

(ii) with F denoting the minimal projector of $\mathcal{C}$ determined by $\langle f; F\rangle = 1$, and $\mathcal{X}_f$ denoting the non-Abelian von Neumann algebra $\{X \in \mathcal{W} \mid [X, F] = 0\}$, then

$$\mathcal{X}_f = \{ Y \in \mathcal{W} \mid \langle \psi; WY \rangle = \langle \psi; YW \rangle$$

$$= \langle \psi; W \rangle \langle \psi; Y \rangle \ \forall W \in \mathcal{W} \}$$

$\mathcal{X}_f \supseteq \mathcal{B}$ for every maximal Abelian von Neumann subalgebra $\mathcal{B}$ of $\mathcal{W}$ with $F \in \mathcal{B}$; $\mathcal{X}_f$ is generated by the collection of all such $\mathcal{B}$'s and the restriction of ψ to $\mathcal{X}_f$ is a normal state which is pure and dispersion-free; (iii) there exists $\Phi \in \mathcal{H}$ such that: (a) $[\mathcal{W}\Phi] = \mathcal{H} = [\mathcal{W}'\Phi]$; (b) with $\Psi = F\Phi / \| F\Phi \|$: $\langle \psi; W \rangle = (W\Psi, \Psi)$ for all $W \in \mathcal{W}$;

$$\text{(c) } \mathcal{C} = \{ A \in \mathcal{W} \mid \langle \phi; [A, W] \rangle = 0 \ \forall W \in \mathcal{W} \}$$

and $\mathcal{C} = \mathfrak{M}_\phi$ where $\langle \phi; W \rangle = (W\Phi, \Phi)$; (d) there exists a faithful normal conditional expectation $\mathcal{E}_f$ from $\mathcal{W}$ onto $\mathcal{X}_f$ such that $\phi \circ \mathcal{E}_f = \phi$; (e) with $\mathcal{S} = [\mathcal{W}\Psi]$, $\mathcal{S}$ is stable under $\mathcal{W}$, and the restriction $\mathcal{W}_\mathcal{S}$ of $\mathcal{W}$ to $\mathcal{S}$ provides a weakly continuous, irreducible representation of the canonical commutation relations.

von Neumann's (1931) uniqueness theorem then ensures that the representation $\mathcal{W}_\mathcal{S}$ just constructed is (unitarily equivalent to) the Schrödinger representation.

The antiunitary, involutive operator J_Φ of the Tomita–Takesaki theory allows one in fact to indicate more precisely how $\mathcal{S}$ sits in $\mathcal{H}$, as the following result shows.

Corollary 4. (i) With $\mathcal{S}$ defined as in Proposition 3, and $\mathcal{T} = J_\Phi \mathcal{S}$, $\mathcal{T}$ is stable under $\mathcal{W}'$, and the restriction $\mathcal{W}_\mathcal{T}$ of $\mathcal{W}'$ to $\mathcal{T}$ provides a weakly continuous, irreducible antirepresentation of the canonical commutation relations; (ii) with $H_\mathcal{S}$ (respectively, $H_\mathcal{T}$) denoting the restriction of H_ϕ (respectively, $J_\Phi H_\phi J_\Phi$) to $\mathcal{S}$ (respectively, $\mathcal{T}$), there exists a unitary map U from $\mathcal{S} \otimes \mathcal{T}$ onto $\mathcal{H}$ such that

$$-(1/\beta) \ln \Delta_\Phi = U(H_\mathcal{S} \otimes I - I \otimes H_\mathcal{T}) U^*$$

Finally, to enhance even more the role of the KMS structures in the present context one can define, following Araki (1980), the operator $S_{\Phi,\Psi}$ by

$$S_{\Phi,\Psi} : W\Phi \in \mathcal{H} \mapsto W^*\Psi \in \mathcal{H}$$

which is again closable; denoting by the same symbol $S_{\Phi,\Psi}$ its closure, one defines the relative modular operator $\Delta_{\Phi,\Psi}=S^*_{\Phi,\Psi}S_{\Phi,\Psi}$ which provides an alternate procedure to recover the objects discussed in Proposition 3 and Corollary 4; one has indeed the following corollary.

Corollary 5. (i) $\mathbb{S}$ (respectively, $\mathcal{T}$) is the closure in $\mathcal{H}$ of the range of $J_\Phi\Delta_{\Phi,\Psi}J_\Phi$ (respectively, $\Delta_{\Phi,\Psi}$); (ii) with Δ^Φ_Ψ (respectively, Λ^Φ_Ψ) denoting the restriction of $J_\Phi\Delta_{\Phi,\Psi}J_\Phi$ (respectively, $\Delta_{\Phi,\Psi}$) to $\mathbb{S}$ (respectively, $\mathcal{T}$):

$$H_{\mathbb{S}}=(1/\beta)\ln\Delta^\Phi_\Psi$$

$$H_{\mathcal{T}}=(1/\beta)\ln\Lambda^\Phi_\Psi$$

2. PROOFS

We start from the von Neumann algebra $\mathcal{W}$ acting on $\mathcal{H}=\mathcal{L}^2(R^{2n},dp\,dq)$ and defined as the weak-operator closure of the linear space spanned by the operators $\{W(z)|z\in C^n\}$, with $z=a+ib$, and

$$[W(z)\Psi](p,q)=\exp[-ib(q-a/2)/\kappa]\Psi(p+b,q-a)$$

One checks indeed easily that with $a,b\in R^n$

$$U(a)=W(a)=\exp(-i\hat{p}a/\kappa)$$

and

$$V(b)=W(ib)=\exp(-i\hat{q}b/\kappa)$$

which satisfy

$$U(a)V(b)=V(b)U(a)\exp(iab/\kappa)$$

so that $\mathcal{W}$ is the von Neumann algebra associated to a representation of the canonical commutation relations on R^n. We now prove that this von Neumann algebra admits at least one cyclic and separating vector. For this purpose, consider with μ and ν positive:

$$\Phi_{\mu\nu}(p,q)=(\mu\nu/\pi^2)^{1/4}\exp[-(\mu p^2+\nu q^2)/2]$$

We have then

$$(U(a)\Phi_{\mu\nu})(p,q)=(\mu\nu/\pi^2)^{1/4}\exp(-\mu p^2/2)\exp[-\nu(q-a)^2/2]$$

Since

$$\overline{\text{span}}\left\{\exp\left[-\mu(p+b)^2/2\right]|b\in R^n\right\}=\mathcal{L}^2(R^n,dp)$$

every square-integrable function of the form $\exp[-\mu p^2/2]f(q)$ can be approximated by linear combinations of elements of the form $[U(a)\Phi_{\mu\nu}](p,q)$. So is it in particular the case for

$$\exp(-\mu p^2/2)\exp(ibq/\kappa)g(q)$$

with g arbitrary in $\mathcal{L}^2(R^n,dq)$.

$$\overline{\text{span}}\left\{\exp\left[-\mu(p+b)^2/2\right]|b\in R^n\right\}=\mathcal{L}^2(R^n,dp)$$

then implies that $[\mathcal{W}\Phi_{\mu\nu}]=\mathcal{H}$, i.e., $\Phi_{\mu\nu}$ is cyclic in $\mathcal{H}$ with respect to $\mathcal{W}$. Upon computing $f_{\mu\nu}(z)=(W(z)\Phi_{\mu\nu},\Phi_{\mu\nu})$ one notices that one can adjust μ and ν in such a manner that $f_{\mu\nu}=f_\Theta$, where

$$f_\Theta(z)=\exp\left[-\Theta(a^2+b^2)/4\kappa\right]$$

in fact we have then $\nu=\Theta/\kappa$ and $\mu=(\Theta^2-1)/\kappa\Theta$; the square-integrability of $\Phi_{\mu\nu}$, i.e., μ, ν positive, implies that one can write without loss of generality

$$\Theta=\coth(\beta\kappa/2)\qquad\text{with}\qquad\beta\in(0,\infty)$$

Let Φ_Θ denote the vector $\Phi_{\mu\nu}$ with these values of μ and ν, and note that

$$(\hat{p}^2\Phi_\Theta,\Phi_\Theta)=(\hat{q}^2\Phi_\Theta,\Phi_\Theta)=\kappa\Theta/2$$

Actually it is easily verified that f_Θ is the canonical equilibrium functional (cf., e.g., Emch, 1972) at the natural temperature β for the harmonic oscillator, and can be written in the Schrödinger representation as

$$f_\Theta(z)=\mathrm{Tr}\,\rho_\Theta W_0(z)$$

with

$$\rho_\Theta=\exp(-\beta H_0)/\mathrm{Tr}\exp(-\beta H_0)$$

where H_0 is the usual harmonic oscillator Hamiltonian in that representation. Consequently ρ_Θ describes a faithful normal state on $\mathcal{B}(\mathcal{H}_0)$ and

$$\phi_\Theta: W\in\mathcal{W}\mapsto(W\Phi_\Theta,\Phi_\Theta)\in C$$

is a faithful normal state on $\mathcal{W}$ which appears thus as an ultraweakly continuous, faithful representation of $\mathcal{B}(\mathcal{H}_0)$. As abstract W^* algebras $\mathcal{W}$ and $\mathcal{B}(\mathcal{H}_0)$ are therefore isomorphic, although they are evidently *not* unitarily equivalent when considered as concrete von Neumann algebras. ϕ_Θ faithful then implies that Φ_Θ is separating for $\mathcal{W}$, and we have thus produced a cyclic separating vector for $\mathcal{W}$ in $\mathcal{H}$.

As a consequence (see, e.g., Dixmier, 1957) of this result, every normal state on $\mathcal{W}$ is a vector state. In particular, for every faithful normal state ϕ on $\mathcal{W}$ there exists a vector Φ in $\mathcal{H}$, which is both cyclic and separating for $\mathcal{W}$ (i.e., $[\mathcal{W}\Phi]=\mathcal{H}=[\mathcal{W}'\Phi]$), such that $\langle\phi; W\rangle=(W\Phi,\Phi)$ for all $W\in\mathcal{W}$.

Since $\mathcal{B}(\mathcal{H}_0)$ is irreducible, it is a factor, i.e., its center consists of the multiples of the identity; this property carries over to $\mathcal{W}$ since the latter, considered as an abstract W^* algebra (with unit!), is isomorphic to $\mathcal{B}(\mathcal{H}_0)$. Hence $\mathcal{W}$ is a factor. An alternate way to prove this, as a consequence of the existence of an ultraweakly continuous isomorphism π_0: $\mathcal{W}\mapsto\mathcal{B}(\mathcal{H}_0)$, is to remark that $\phi_\Theta\circ\pi_0^{-1}$ is the only (normal) KMS state on $\mathcal{B}(\mathcal{H}_0)$ for the natural temperature β and the evolution corresponding to the harmonic oscillator. ϕ_Θ is therefore extremal KMS and consequently $\mathcal{W}$ is a factor.

This concludes the proof of Proposition 1. Note that the harmonic oscillator only played a role as a convenient intermediary step in the proof: the proposition itself has indeed been shown to hold for every faithful normal state ϕ on $\mathcal{W}$.

With ϕ any faithful, nondegenerate, normal state on $\mathcal{W}$ and π_0 an ultraweakly continuous isomorphism from $\mathcal{W}$ onto $\mathcal{B}(\mathcal{H}_0)$ we have that $\phi\circ\pi_0^{-1}$ is a faithful, nondegenerate, normal state on $\mathcal{B}(\mathcal{H}_0)$ and we can therefore write

$$\langle\phi; W\rangle=\operatorname{Tr}\rho_\phi\pi_0(W)\qquad\forall\qquad W\in\mathcal{W}$$

with

$$\rho_\phi=\sum_k\lambda_k\pi_0(P_k)$$

where

$$\{\pi_0(P_k)|k\in Z^+\}$$

is a partition of the identity on $\mathcal{H}_0$ into mutually orthogonal one-dimensional projectors,

$$\lambda_k\in(0,1)\qquad\forall\qquad k\in Z^+$$

and

$$\sum_k \lambda_k = 1; \qquad k \neq l \text{ implies } \lambda_k \neq \lambda_l$$

We can now define $H_\phi \in \mathcal{W}$ by $\pi_0(H_\phi) = -(1/\beta)\ln \rho_\phi$. One then checks easily that ϕ satisfies the KMS condition at the natural temperature β, with respect to the evolution

$$\alpha_\phi(t)[W] = e^{iH_\phi t} W e^{-iH_\phi t}$$

and that H_ϕ is defined uniquely, up to an additive constant, by this condition, as a consequence of $\mathcal{W} \cap \mathcal{W}' = C \cdot I$.

Since the spectrum of $\pi_0(H_\phi)$ is discrete and nondegenerate, we have $\{\pi_0(H_\phi)\}' = \{\pi_0(H_\phi)\}''$, and thus the centralizer $\mathfrak{M}_\phi$ of $\mathcal{W}$ is an atomic, maximal Abelian von Neumann subalgebra of $\mathcal{W}$. This proves parts (i), (ii), (iiic), and (iiid) of Proposition 2.

To continue following as explicit a path as is possible, we notice that $\mathcal{H} = \mathcal{L}^2(R^{2n}, dp\,dq)$ is naturally isomorphic to the space of Hilbert–Schmidt operators on $\mathcal{H}_0 = \mathcal{L}^2(R^n, dx)$. Let then $\{\Phi_k | k \in Z^+\}$ be the orthonormal basis in $\mathcal{H}_0$ determined by $\pi_0(P_k)\Phi_l = \delta_{kl}\Phi_l$; let further Φ_{kl} be the corresponding orthonormal basis in $\mathcal{H}$. One then verifies easily that

$$\Phi = \sum_k \lambda_k^{1/2} \Phi_{kk}$$

$$\Delta_\Phi \Phi_{kl} = (\lambda_k / \lambda_l) \Phi_{kl}$$

$$J_\Phi \Phi_{kl} = \Phi_{lk}$$

from which the remainder of Proposition 2 follows by a straightforward computation.

Conversely, let $\mathcal{Q}$ and f be as in Proposition 3. Since π_0 is a normal isomorphism, $f \circ \pi_0^{-1}$ is a pure, normal state on $\pi_0(\mathcal{Q})$ which is an atomic, maximal Abelian von Neumann subalgebra of $\mathfrak{B}(\mathcal{H}_0) = \pi_0(\mathcal{W})$. Moreover $\mathcal{Q}$ determines, via $\mathcal{Q} = \{P_k | k \in Z^+\}''$ a partition of the identity on $\mathcal{H}_0$ into mutually orthogonal projectors $\{\pi_0(P_k) | k \in Z^+\}$. Since $f \circ \pi_0^{-1}$ is normal, there exists at least some k for which $\langle f; P_k \rangle \neq 0$; and since f is pure, there exists exactly one P_k, which we write F, such that $\langle f; P_k \rangle = 1$. By the Hahn–Banach theorem, there exists at least one extension of $f \circ \pi_0^{-1}$ to a state ω on $\mathfrak{B}(H_0)$, and this state can be written uniquely (see for instance Dixmier, 1957) as $\omega = \omega_1 + \omega_2$ with ω_1 ultraweakly continuous and ω_2 annihilating the compact operators on $\mathcal{H}_0$; moreover $\|\omega\| = \|\omega_1\| + \|\omega_2\|$. In particular

$$\langle \omega_1 \circ \pi_0; F \rangle = \langle \omega \circ \pi_0; F \rangle = \langle f; F \rangle = 1$$

and thus $\|\omega_1\| - 1$ positive. Since however ω is a state, $\|\omega\| = 1$ and thus $\|\omega_2\| = 0 = \omega_2$, i.e., ω is normal. It is then easily checked that ω is pure and is uniquely determined by its restriction to $\pi_0(\mathcal{Q})$. This part of the argument can be found already in Kadison and Singer (1959) where counterexamples are exhibited to show the importance of the discreteness of $\mathcal{Q}$ [for the role of normality and an up-dated discussion of the general extension problem, see also Anderson (1980) and references quoted therein]. Upon using the isomorphism π_0, we conclude that $\psi = \omega \circ \pi_0^{-1}$ is the unique state extension of f to $\mathcal{W}$, and that it is pure and normal, thus proving part (i) of Proposition 3. Part (ii) of that proposition is then again verified when one reads it, through π_0, as a statement in $\mathcal{B}(\mathcal{H}_0)$. The faithful normal state ϕ of part (iii) is obtained as follows. Choose $\{\varepsilon_k | k \in Z^+\} \subseteq R$ such that $\varepsilon_k \neq \varepsilon_l$ whenever $k \neq l$, and $\Sigma_k \exp(-\beta\varepsilon_k)$ finite; form then $H = \Sigma_k \varepsilon_k P_k$ and

$$\rho = \exp[-\beta\pi_0(H)] / \mathrm{Tr} \exp[-\beta\pi_0(H)]$$

and define ϕ by $\langle \phi; W \rangle = \mathrm{Tr}\, \rho\, \pi_0(W)$. Since this state is faithful, nondegenerate, and normal, we can use Proposition 1 to assert the existence of Φ satisfying (iiia). Part (iiib) of Proposition 3 then follows from

$$\langle \psi; W \rangle = \langle \phi; FWF \rangle / \langle \phi; F \rangle \qquad \forall \qquad W \in \mathcal{W}$$

Part (iiic) asserts, in agreement with Proposition 2, that $\mathcal{Q} = \mathcal{M}_\phi$, a fact which is easily verified when one reads this, via π_0, as a statement in $\mathcal{B}(\mathcal{H}_0)$. Since $\mathcal{X}_f$ is stable under the modular group $\alpha_\phi(R)$, we know from Takesaki (1972) that there exists a faithful normal conditional expectation, i.e., an ultraweakly continuous projection $\mathcal{E}_f$ of norm 1, from $\mathcal{W}$ onto $\mathcal{X}_f$, such that $\phi \circ \mathcal{E}_f = \phi$. Finally ψ pure and normal on $\mathcal{W}$ implies (iiie), thus completing the proof of Proposition 3.

The two corollaries are easily derived as consequences of the main propositions, upon using the isomorphisms described in the above proofs. In particular, the relative modular operator $\Delta_{\Phi\Psi}$ is given explicitly, in the basis introduced in the proof of Proposition 2, by

$$\Delta_{\Phi\Psi}\Phi_{kl} = \lambda_l^{-1}\delta_{k0}\Phi_{0l}$$

3. CONCLUDING REMARKS AND SPECULATIONS

(1) The elementary character of the proofs presented in Section 2 emphasizes the immediacy of the KMS structures underlying the primary (respectively, irreducible) representations $\mathcal{W}$ (respectively, $\mathcal{W}_{\mathcal{S}}$) of the canonical commutation relations obtained from the prequantization procedure. No claim is laid on the originality of the separate ingredients appearing in these proofs: much of these has been adapted from the fundamental papers on the algebraic formulation of the KMS condition. The point rather was to use as simple a language as possible to bring these structures in contact with another area of mathematical physics: the geometric quantization program.

(2) The appearance of concepts from statistical mechanics (namely, the KMS condition and, through it, the natural temperature β) in the description of $\mathcal{W}$ should be compared to the fact that the positive smooth functions of $\mathcal{C}$ appear in classical mechanics not only as observables, and as generators (via the Poisson bracket) of one-parameter groups of diffeomorphisms, but also as Radon–Nikodym derivatives of measures which are absolutely continuous with respect to the measure $dp\,dq$ associated to the two-form ω. These smooth measures play in classical mechanics the same role as that played in quantum mechanics by the normal states on $\mathcal{W}$. The special role played by the faithful normal states should thus be compared to that played by the smooth measures whose null sets coincide with the null sets of $dp\,dq$, i.e., the measures of the form $\exp[-\beta f(p,q)]dp\,dq$.

(3) The fact that $\mathcal{H}$ harbors operator realizations of both classical mechanics (namely, $\hat{\mathcal{C}}$) and quantum mechanics (namely, $\mathcal{W}$) illustrates the seminal argument originally put forward by Born and Jordan (1925) that the Heisenberg (1925) formulation of quantum mechanics involves changing the rules by which one forms functions, and in particular products and even powers, of observables.

(4) The splitting of $\mathcal{H}$ into $\mathcal{S}\otimes\mathcal{T}$ and the attendant extraction of the irreducible representation $\mathcal{W}_{\mathcal{S}}$ from the primary representation $\mathcal{W}$, based as it is on a choice of an atomic, maximal Abelian von Neumann subalgebra $\mathcal{Q}$ of $\mathcal{W}$, should be compared to the Lagrangian foliation (see Kostant, 1970; Souriau, 1966; Weinstein, 1971), which plays a central role in the geometric quantization programme (see, e.g., Guillemin and Sternberg, 1977; Simms and Woodhouse, 1976; Sniatycki, 1980).

One should also note the simultaneous presence, together with the irreducible representation $\mathcal{W}_{\mathcal{S}}$, of the irreducible antirepresentation $\mathcal{W}_{\mathcal{T}} = J\mathcal{W}_{\mathcal{S}}J$ which we extracted from $\mathcal{W}'$. The KMS structures delineated in the present paper thus also add some new perspective to the investigations reported in Segal (1960), Klauder (1964), Streater (1966) (and thus Bargmann, 1961).

One might further want to explore the connection between these structures and the work of Borchers (1973). Indeed the representation $\mathcal{W}$ of the CCR, discussed in the present paper as pertaining to the prequantization program, also coincide with the representation of the CCR obtained by constructing the regular representation of the cross-product (see, e.g., Pedersen, 1979) associated to the dynamical system $\{\mathcal{Q}, R^n, \alpha\}$ where $\mathcal{Q}$ is the von Neumann algebra generated by the operators $\{V_0(b)|b\in R^n\}$ defined by

$$(V_0(b)\Psi)(p)=\Psi(p+b)$$

acting on $\mathcal{H}_0=\mathcal{L}^2(R^n, dp)$; and α is the homomorphism $R^n\mapsto\text{Aut}(\mathcal{Q})$ defined by

$$\alpha_a[V_0(b)]=\exp(iab/\kappa)V_0(b)$$

We have indeed

$$\mathcal{H}=\left\{\Psi: R^n\mapsto\mathcal{H}_0\left|\int dq\,\|\Psi(q)\|^2_{\mathcal{H}_0}\right|\text{ finite}\right\}$$

$$[V(b)\Psi](q)=\alpha_{-q}[V_0(b)]\Psi(q)$$

$$[U(\alpha)\Psi](q)=\Psi(q-a)$$

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